DTP/00/76, IPPP/00/11, MADPH-00-1206, hep-ph/0012007

One-loop QCD corrections to quark scattering at NNLO*

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ABSTRACT: We present the $\mathcal{O}(\alpha_s^4)$ virtual QCD corrections to unlike-quark $q\bar{q} \to q'\bar{q}'$ and like-quark scattering $q\bar{q} \to q\bar{q}$ due to the interference of one-loop amplitudes with one-loop amplitudes. The structure of the infrared divergences agrees with that predicted by Catani. The results are expressed in an analytic form so that the relevant expressions for crossed scattering processes can be obtained in a straightforward manner. The one-loop contributions presented here, together with the interference of tree with two-loop amplitudes given recently, complete the $2 \to 2$ virtual QCD corrections at NNLO for the massless quark scattering processes.

KEYWORDS: QCD, Jets, LEP HERA and SLC Physics, NLO and NNLO Computations.

*Work supported in part by the UK Particle Physics and Astronomy Research Council and by the EU Fourth Framework Programme 'Training and Mobility of Researchers', Network 'Quantum Chromodynamics and the Deep Structure of Elementary Particles', contract FMRX-CT98-0194 (DG 12 - MIHT). C.A. acknowledges the financial support of the Greek government and M.E.T. acknowledges financial support from CONACyT and the CVCP. We thank the British Council and German Academic Exchange Service for support under ARC project 1050.

1. Introduction

The increasing precision of high energy scattering experiments demonstrates the need for very precise theoretical calculations. The accuracy of existing next-to-leading-order (NLO) predictions may be improved by including the next highest term in the perturbation series. Such next-to-next-to-leading-order (NNLO) estimates will improve the theoretical precision in two ways. First, the renormalisation scale dependence will decrease (from about 10% for NLO predictions for central production of a jet with transverse energy $E_T \sim 100$ GeV at the Tevatron to approximately $1-2\%$ at NNLO). Second, better matching of the parton level theoretical and hadron level experimental jet algorithms will be possible since at NNLO, three partons can combine to form the jet rather than two.

Recent progress towards two-loop integrals [1]–[8] has turned the calculation of NNLO virtual corrections for massless $2 \to 2$ processes into a viable task. Bern, Dixon and Kosower [9] were the first to address such scattering processes and provided analytic expressions for the maximal-helicity-violating two-loop amplitude for $gg \to gg$. More recently, Bern, Dixon and Ghinculov [10] completed the two-loop calculation of physical $2 \to 2$ scattering amplitudes for the QED processes $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to e^-e^+$.

Subsequently, we have derived analytical expressions for the $\mathcal{O}(\alpha_s^4)$ two-loop contribution to unlike quark scattering $q\bar{q} \to q'\bar{q}'$ [11], and like quark scattering $q\bar{q} \to q\bar{q}$ [12], as well as the crossed and time reversed processes, in the limit where the quark mass can be neglected. To complete the calculation of the NNLO virtual corrections, the interference terms of one-loop amplitudes with one-loop amplitudes need to be included. It is the purpose of this paper to present analytical expressions for these contributions using conventional dimensional regularisation (space-time dimension $D = 4 - 2\epsilon$), renormalised with the $\overline{\mathrm{MS}}$ scheme.

Our paper is organised as follows. We establish our notation in Section 2 and briefly describe our method in Section 3. In Sections 4.1 and 4.2, we provide analytic expressions of the one-loop contributions at NNLO for the unlike and like-quark scattering respectively, obtained by direct evaluation of the Feynman diagrams. Our results are expressed in terms of two master integrals, the one-loop box graph in $6-2\epsilon$ dimensions and the one-loop bubble graph in $4 - 2\epsilon$ dimensions. A clear separation of the infrared poles is apparent by inspection of the results. We find complete agreement between the infrared structure obtained by our explicit calculation and that anticipated by Catani [13]. Analytic expansions in ϵ for all kinematic regions are straightforward to derive by inserting the expansions of the master integrals in the appropriate region. In Section 5 we summarize our results.

2. Notation

We follow the notation of Refs. [11, 12] as closely as possible and consider the unlike-quark scattering process

$$q(p_1) + \bar{q}(p_2) + q'(p_3) + \bar{q}'(p_4) \to 0, \tag{2.1}$$

and the like-quark scattering process

$$q(p_1) + \bar{q}(p_2) + q(p_3) + \bar{q}(p_4) \to 0, \tag{2.2}$$

where particles are incoming and carry light-like momenta (shown in parentheses). Their total momentum is conserved, satisfying

$$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0,$$

and the associated Mandelstam variables are given by

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2, \qquad u = (p_1 + p_3)^2. \tag{2.3}$$

We use conventional dimensional regularisation and treat the external quark states in D space-time dimensions and renormalise the ultraviolet divergences in the $\overline{\mathrm{MS}}$ scheme. The bare coupling α_0 is related to the running coupling $\alpha_s \equiv \alpha_s(\mu^2)$, at renormalisation scale μ, by

$$\alpha_0 \, S_\epsilon = \alpha_s \left[1 - \frac{\beta_0}{\epsilon} \left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right) \left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}(\alpha_s^3) \right], \tag{2.4}$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma}, \qquad \gamma = 0.5772\ldots = \text{Euler constant}, \tag{2.5}$$

is the typical phase-space volume factor in $D = 4 - 2\epsilon$ dimensions. As usual, the first two coefficients of the QCD beta function, β_0 and β_1 for N_F (massless) quark flavours are

$$\beta_0 = \frac{11 C_A - 4 T_R N_F}{6}, \qquad \beta_1 = \frac{17 C_A^2 - 10 C_A T_R N_F - 6 C_F T_R N_F}{6}. \tag{2.6}$$

where N is the number of colours, and

$$C_F = \frac{N^2 - 1}{2N}, \qquad C_A = N, \qquad T_R = \frac{1}{2}. \tag{2.7}$$

The renormalised amplitude for the unlike-quark process is given by

$$|\mathcal{M}\rangle_{unlike} = 4\pi\alpha_s \left[|\mathcal{M}^{(0)}\rangle + \left(\frac{\alpha_s}{2\pi}\right) |\mathcal{M}^{(1)}\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2 |\mathcal{M}^{(2)}\rangle + \mathcal{O}\left(\alpha_s^3\right) \right], \tag{2.8}$$

with $|\mathcal{M}^{(i)}\rangle$ representing the i-loop amplitude in colour-space. For the like-quarks we have the related expression

$$|\mathcal{M}\rangle_{like} = 4\pi\alpha_s\Bigg[\left(|\mathcal{M}^{(0)}\rangle - |\overline{\mathcal{M}}^{(0)}\rangle\right) + \left(\frac{\alpha_s}{2\pi}\right)\left(|\mathcal{M}^{(1)}\rangle - |\overline{\mathcal{M}}^{(1)}\rangle\right) + \left(\frac{\alpha_s}{2\pi}\right)^2\left(|\mathcal{M}^{(2)}\rangle - |\overline{\mathcal{M}}^{(2)}\rangle\right) + \mathcal{O}(\alpha_s^3)\Bigg]. \quad (2.9)$$

Here $|\overline{\mathcal{M}}^{(i)}\rangle$ describes the t-channel graphs which can be obtained from the s-channel diagrams by exchanging the roles of particles 2 and 4

$$|\overline{\mathcal{M}}^{(i)}\rangle = |\mathcal{M}^{(i)}\rangle(2 \leftrightarrow 4). \quad (2.10)$$

Both $|\mathcal{M}^{(i)}\rangle$ and $|\overline{\mathcal{M}}^{(i)}\rangle$ are renormalisation scale and renormalisation scheme dependent.

In squaring the amplitudes and summing over colours and spins we find two types of terms,

- the self-interference of the graphs in a single channel, described by the function $\mathcal{A}(s,t,u)$ for the s-channel and $\mathcal{A}(t,s,u)$ for the t-channel, and
- the interference of the s-channel graphs with the t-channel graphs, described by the function $\mathcal{B}(s,t,u)$.

Thus, for distinct quark scattering we have

$$\langle\mathcal{M}|\mathcal{M}\rangle_{unlike} = \sum|\mathcal{M}(q+\bar{q} \to \bar{q}'+q')|^2 = \mathcal{A}(s,t,u), \quad (2.11)$$

while for identical quarks

$$\begin{aligned}\langle\mathcal{M}|\mathcal{M}\rangle_{like} &= \sum|\mathcal{M}(q+\bar{q} \to \bar{q}+q)|^2 \\ &= \mathcal{A}(s,t,u) + \mathcal{A}(t,s,u) + \mathcal{B}(s,t,u). \end{aligned} \quad (2.12)$$

Similarly, for the crossed and time-reversed processes we obtain

$$\sum|\mathcal{M}(q+q' \to q+q')|^2 = \mathcal{A}(u,t,s) \quad (2.13)$$
$$\sum|\mathcal{M}(q+\bar{q}' \to q+\bar{q}')|^2 = \mathcal{A}(t,s,u) \quad (2.14)$$
$$\sum|\mathcal{M}(\bar{q}+\bar{q}' \to \bar{q}+\bar{q}')|^2 = \mathcal{A}(u,t,s) \quad (2.15)$$
$$\sum|\mathcal{M}(q+\ q \to q+\ q)|^2 = \mathcal{A}(u,t,s) + \mathcal{A}(t,u,s) + \mathcal{B}(u,t,s). \quad (2.16)$$

The function $\mathcal{A}$ can be expanded perturbatively to yield

$$\mathcal{A}(s,t,u) = 16\pi^2\alpha_s^2\left[\mathcal{A}^4(s,t,u) + \left(\frac{\alpha_s}{2\pi}\right)\mathcal{A}^6(s,t,u) + \left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{A}^8(s,t,u) + \mathcal{O}(\alpha_s^3)\right], \quad (2.17)$$

where

$$\mathcal{A}^4(s,t,u) = \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle \equiv 2(N^2-1)\left(\frac{t^2+u^2}{s^2} - \epsilon\right), \tag{2.18}$$

$$\mathcal{A}^6(s,t,u) = \Big(\langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(1)} | \mathcal{M}^{(0)} \rangle \Big), \tag{2.19}$$

$$\mathcal{A}^8(s,t,u) = \Big(\langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle \Big). \tag{2.20}$$

In the same manner

$$\mathcal{B}(s,t,u) = 16\pi^2\alpha_s^2 \left[\mathcal{B}^4(s,t,u) + \left(\frac{\alpha_s}{2\pi}\right)\mathcal{B}^6(s,t,u) + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{B}^8(s,t,u) + \mathcal{O}(\alpha_s^3) \right], \tag{2.21}$$

where, in terms of the amplitudes, we have

$$\begin{aligned}\mathcal{B}^4(s,t,u) &= -\left(\langle \overline{\mathcal{M}}^{(0)} | \mathcal{M}^{(0)} \rangle + \langle \mathcal{M}^{(0)} | \overline{\mathcal{M}}^{(0)} \rangle \right) \\ &\equiv -4\left(\frac{N^2-1}{N}\right)(1-\epsilon)\left(\frac{u^2}{st} + \epsilon\right),\end{aligned} \tag{2.22}$$

$$\mathcal{B}^6(s,t,u) = -\left(\langle \overline{\mathcal{M}}^{(1)} | \mathcal{M}^{(0)} \rangle + \langle \mathcal{M}^{(0)} | \overline{\mathcal{M}}^{(1)} \rangle + \langle \overline{\mathcal{M}}^{(0)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(1)} | \overline{\mathcal{M}}^{(0)} \rangle \right) \tag{2.23}$$

$$\begin{aligned}\mathcal{B}^8(s,t,u) = -\Big(&\langle \overline{\mathcal{M}}^{(1)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(1)} | \overline{\mathcal{M}}^{(1)} \rangle \\ &+ \langle \overline{\mathcal{M}}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \overline{\mathcal{M}}^{(0)} \rangle + \langle \mathcal{M}^{(0)} | \overline{\mathcal{M}}^{(2)} \rangle + \langle \overline{\mathcal{M}}^{(2)} | \mathcal{M}^{(0)} \rangle \Big).\end{aligned} \tag{2.24}$$

Expressions for $\mathcal{A}^6$ and $\mathcal{B}^6$, valid in dimensional regularisation, are given in Ref. [14]. Analytical formulae for the two-loop contribution to $\mathcal{A}^8$

$$\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle$$

are given in Ref. [11], with analogous expressions for the two-loop contribution to $\mathcal{B}^8$

$$\langle \overline{\mathcal{M}}^{(0)} | \mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)} | \overline{\mathcal{M}}^{(0)} \rangle + \langle \mathcal{M}^{(0)} | \overline{\mathcal{M}}^{(2)} \rangle + \langle \overline{\mathcal{M}}^{(2)} | \mathcal{M}^{(0)} \rangle$$

given in Ref. [12].

We now concentrate on the contributions to both $\mathcal{A}^8$ and $\mathcal{B}^8$ due to the interference of one-loop amplitudes with one-loop amplitudes, namely

$$\mathcal{A}^{8\,(1\times1)}(s,t,u) = \langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle, \tag{2.25}$$

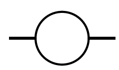
and

$$\mathcal{B}^{8\,(1\times1)}(s,t,u) = -\left(\langle \overline{\mathcal{M}}^{(1)} | \mathcal{M}^{(1)} \rangle + \langle \mathcal{M}^{(1)} | \overline{\mathcal{M}}^{(1)} \rangle \right). \tag{2.26}$$

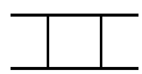
Even though they are somewhat simpler to evaluate than the two loop graphs, they form a vital part of the NNLO virtual corrections and we present them here for completeness. One-loop helicity amplitudes for the $2 \to 2$ quark scattering processes were given in Ref. [15] as truncated expansions in ϵ including their finite part. However, this is only sufficient to obtain the pole structure of $\mathcal{A}^{8\,(1\times1)}$ and $\mathcal{B}^{8\,(1\times1)}$ up to $1/\epsilon^2$. To determine the $1/\epsilon$ and finite parts requires knowledge of the one-loop amplitude through to $\mathcal{O}(\epsilon^2)$.

3. Method

We organise our calculation as follows. First the one-loop Feynman diagrams are generated using `QGRAF` [16]. The emerging tensor integrals are associated with scalar integrals in higher dimension and with higher powers of propagators [6, 17]. Systematic application of the integration-by-parts identities [18] is sufficient to reduce these higher-dimension, higher-power integrals to master integrals in $D = 4-2\epsilon$. We can express all the integrals of the one-loop amplitudes in terms of just two master integrals, the scalar bubble graph

$$\mathrm{Bub}(s) =$$  $$(s) \quad ,$$

and the one-loop scalar box graph

$$\mathrm{Box}(s,t) =$$  $$(s,t).$$

The above choice of master integrals is not unique. We prefer to replace the one-loop box in $D = 4 - 2\epsilon$ by the finite one-loop box in $D = 6 - 2\epsilon$, Box^6. This leads to a natural separation of the infrared poles and the finite part of the one-loop amplitudes. As a last step we multiply together the one-loop amplitudes and perform the colour and Dirac traces.

4. Results

In this section we give explicit formulae for both $\mathcal{A}^{8\,(1\times1)}(s,t,u)$ and $\mathcal{B}^{8\,(1\times1)}(s,t,u)$, in terms of the finite Box^6 and the $1/\epsilon$ divergent Bub master integrals. The finite parts depend only on Box^6 and differences of the Bub master integrals.

4.1 Unlike quarks

In the unlike-quark case we obtain,

$$\begin{aligned}\mathcal{A}^{8\,(1\times1)}(s,t,u) = {} & \left[|\mathcal{IR}_t + \mathcal{F}_r + \mathcal{F}_g|^2 + \left(N^2-1\right)|\mathcal{IR}_{nt}|^2\right]\langle\mathcal{M}_0|\mathcal{M}_0\rangle \\ & + 2\,\mathrm{Re}\left[\left(\mathcal{IR}_t + \mathcal{F}_r + \mathcal{F}_g\right)^\dagger \mathcal{F}_1 + \left(N^2-1\right)\mathcal{IR}_{nt}^\dagger \mathcal{F}_2\right]\end{aligned}$$

$$
+ \left(N^2-1\right)\left[\frac{N^4-3N^2+3}{N^2}\mathcal{F}_3(s,t,u)+\frac{N^2+3}{N^2}\mathcal{F}_3(s,u,t)\right.
$$
$$
\left.+\frac{N^2-3}{N^2}\left[\mathcal{F}_4(s,t,u)+\mathcal{F}_4(s,u,t)\right]\right], \tag{4.1}
$$

where the infrared poles present in the one-loop amplitude proportional to the tree-level matrix elements are given by

$$
\mathcal{IR}_t = \frac{2}{\epsilon(2+\epsilon)}\left[\frac{1}{N}\mathrm{Bub}(s)-\frac{2}{N}\mathrm{Bub}(u)-\frac{(N^2-2)}{N}\mathrm{Bub}(t)\right], \tag{4.2}
$$
$$
\mathcal{IR}_{nt} = \frac{2}{\epsilon(2+\epsilon)}\left[\frac{1}{N}\mathrm{Bub}(u)-\frac{1}{N}\mathrm{Bub}(t)\right], \tag{4.3}
$$

which diverge as $1/\epsilon^2$ and $1/\epsilon$ respectively. Both

$$
\mathcal{F}_r = \beta_0\left(-\frac{1}{\epsilon}+\frac{3(1-\epsilon)}{3-2\epsilon}\ \mathrm{Bub}(s)\right), \tag{4.4}
$$

and

$$
\mathcal{F}_g = \frac{\epsilon\left[N^2(11+2\epsilon)+9-4\epsilon^2\right]}{2(2+\epsilon)(3-2\epsilon)N}\ \mathrm{Bub}(s), \tag{4.5}
$$

are finite terms multiplying the tree-level matrix elements. The functions

$$
\mathcal{F}_1 = \frac{N^2-1}{2N}\left[(N^2-2)f(s,t,u)+2f(s,u,t)\right], \tag{4.6}
$$

and

$$
\mathcal{F}_2 = \frac{N^2-1}{2N}\left[f(s,t,u)-f(s,u,t)\right] \tag{4.7}
$$

are finite and multiplied by the infrared poles of the conjugated one-loop amplitude, with

$$
f(s,t,u) = \left[\frac{3s^2+7u^2+9t^2}{s^2}-4\frac{u^2+t^2+2s^2}{(2+\epsilon)s^2}+\epsilon\frac{5u+7t}{s}\right]\left[\mathrm{Bub}(t)-\mathrm{Bub}(s)\right]
$$
$$
+u(1-2\epsilon)\frac{6t^2+2u^2-3\epsilon s^2}{s^2}\ \mathrm{Box}^6(s,t). \tag{4.8}
$$

Finally the square of the finite part of the one-loop amplitude is fixed by the finite functions $\mathcal{F}_3$ and $\mathcal{F}_4$,

$$
\mathcal{F}_3(s,t,u) = \left|\mathrm{Box}^6(s,t)\right|^2\left[\frac{t^4+6t^2u^2+u^4}{2s^2}\right]
$$
$$
+2\,\mathrm{Re}\left\{\left[\mathrm{Bub}(t)-\mathrm{Bub}(s)\right]^\dagger\ \mathrm{Box}^6(s,t)\right\}\left[\frac{2u^3-tu^2+8t^2u-t^3}{2s^2}\right]
$$
$$
+\left|\mathrm{Bub}(t)-\mathrm{Bub}(s)\right|^2\left[\frac{5t^2-2tu+2u^2}{s^2}\right]+\mathcal{O}(\epsilon), \tag{4.9}
$$

and

$$\mathcal{F}_4(s,t,u) = 2\,\mathrm{Re}\left\{\mathrm{Box}^{6\dagger}(s,t)\mathrm{Box}^6(s,u)\right\}\left[\frac{tu(t^2+u^2)}{s^2}\right]$$
$$+2\,\mathrm{Re}\left\{[\mathrm{Bub}(u)-\mathrm{Bub}(s)]^\dagger\ \mathrm{Box}^6(s,t)\right\}\left[\frac{u(7t^2-2tu+3u^2)}{2s^2}\right]$$
$$+2\,\mathrm{Re}\left\{[\mathrm{Bub}(u)-\mathrm{Bub}(s)]^\dagger\,[\mathrm{Bub}(t)-\mathrm{Bub}(s)]\right\}\left[\frac{3(t^2-tu+u^2)}{2s^2}\right]+\mathcal{O}(\epsilon). \tag{4.10}$$

In the latter expressions, we have discarded contributions of $\mathcal{O}(\epsilon)$.

After explicit series expansion in ϵ, the infrared singular terms $\mathcal{IR}_t$ and $\mathcal{IR}_{nt}$ reproduce the pole structure obtained by expanding

$$\mathcal{IR}_{t,C} = \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\left[\frac{1}{N}\left(-\frac{\mu^2}{s}\right)^\epsilon-\frac{2}{N}\left(-\frac{\mu^2}{u}\right)^\epsilon-\frac{(N^2-2)}{N}\left(-\frac{\mu^2}{t}\right)^\epsilon\right], \tag{4.11}$$

$$\mathcal{IR}_{nt,C} = \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\left[\frac{1}{N}\left(-\frac{\mu^2}{u}\right)^\epsilon-\frac{1}{N}\left(-\frac{\mu^2}{t}\right)^\epsilon\right], \tag{4.12}$$

which is the singular structure obtained by straightforward application of the formalism of [13]. To rewrite Eq. (4.1) directly in terms of $\mathcal{IR}_{t,C}$ and $\mathcal{IR}_{nt,C}$ rather than $\mathcal{IR}_t$ and $\mathcal{IR}_{nt}$ requires the finite difference to be evaluated through to $\mathcal{O}(\epsilon^2)$.

Equation (4.1) is valid in all kinematic regions. Series expansions in ϵ in a particular region can be easily obtained by inserting the appropriate expansions of the master integrals. In this equation, the finite functions are multiplied by poles in ϵ, so they must be expanded through to $\mathcal{O}(\epsilon^2)$. In the physical region $u<0$, $t<0$, $\mathrm{Box}^6(u,t)$ has no imaginary part and is given by [10]

$$\mathrm{Box}^6(u,t) = \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{2s\Gamma(2-2\epsilon)}\left(-\frac{\mu^2}{s}\right)^\epsilon\times\left[\frac{1}{2}((L_x-L_y)^2+\pi^2)\right.$$
$$+2\epsilon\left(\mathrm{Li}_3(x)-L_x\mathrm{Li}_2(x)-\frac{1}{3}L_x^3-\frac{\pi^2}{2}L_x\right)$$
$$-2\epsilon^2\left(\mathrm{Li}_4(x)+L_y\mathrm{Li}_3(x)-\frac{1}{2}L_x^2\mathrm{Li}_2(x)-\frac{1}{8}L_x^4-\frac{1}{6}L_x^3L_y+\frac{1}{4}L_x^2L_y^2\right.$$
$$\left.\left.-\frac{\pi^2}{4}L_x^2-\frac{\pi^2}{3}L_xL_y-\frac{\pi^4}{45}\right)+(u\leftrightarrow t)\right]+\mathcal{O}(\epsilon^3), \tag{4.13}$$

where $x=-t/s$, $L_x=\log(x)$ and $L_y=\log(1-x)$ and the polylogarithms $\mathrm{Li}_n(z)$ are defined by,

$$\mathrm{Li}_n(z) = \int_0^z\frac{dt}{t}\mathrm{Li}_{n-1}(t)\qquad\text{for } n=2,3,4 \tag{4.14}$$

$$\mathrm{Li}_2(z) = -\int_0^z\frac{dt}{t}\log(1-t). \tag{4.15}$$

Analytic continuation to other kinematic regions is obtained using the inversion formulae for the arguments of the polylogarithms (see for example [6]) when $x > 1$,

$$
\begin{aligned}
\mathrm{Li}_2(x+i0) &= -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{1}{2}\log^2(x) + \frac{\pi^2}{3} + i\pi\log(x) \\
\mathrm{Li}_3(x+i0) &= \mathrm{Li}_3\left(\frac{1}{x}\right) - \frac{1}{6}\log^3(x) + \frac{\pi^2}{3}\log(x) + \frac{i\pi}{2}\log^2(x) \\
\mathrm{Li}_4(x+i0) &= -\mathrm{Li}_4\left(\frac{1}{x}\right) - \frac{1}{24}\log^4(x) + \frac{\pi^2}{6}\log^2(x) + \frac{\pi^4}{45} + \frac{i\pi}{6}\log^3(x) \quad (4.16)
\end{aligned}
$$

Finally, the one-loop bubble integral in $D = 4 - 2\epsilon$ dimensions is given by,

$$
\mathrm{Bub}(s) = \frac{e^{\epsilon\gamma}\Gamma\left(1+\epsilon\right)\Gamma\left(1-\epsilon\right)^2}{\Gamma\left(2-2\epsilon\right)\epsilon}\left(-\frac{\mu^2}{s}\right)^{\epsilon}, \tag{4.17}
$$

and can be easily expanded in ϵ in all kinematic regions. Note that the complex conjugate of the master integrals is also required in Eq. (4.1).

4.2 Like quarks

For the like-quark contribution we find a similar expression,

$$
\begin{aligned}
&\mathcal{B}^{8\,(1\times 1)}(s,t,u) = \\
&\quad -2\,\mathrm{Re}\Bigg\{ \left(\overline{\mathcal{IR}}_t + \overline{\mathcal{F}}_r + \overline{\mathcal{F}}_g\right)^\dagger \left(\mathcal{IR}_t + \mathcal{F}_r + \mathcal{F}_g\right) \langle\overline{\mathcal{M}}_0|\mathcal{M}_0\rangle \\
&\qquad +(N^2-1)\left(\overline{\mathcal{IR}}_{nt} - \overline{\mathcal{F}}_r - \overline{\mathcal{F}}_g\right)^\dagger \mathcal{IR}_{nt}\,\langle\overline{\mathcal{M}}_0|\mathcal{M}_0\rangle \\
&\qquad + \left[\left(\overline{\mathcal{IR}}_t + \overline{\mathcal{F}}_r + \overline{\mathcal{F}}_g\right)^\dagger \mathcal{F}_1' + (N^2-1)\left(\overline{\mathcal{IR}}_{nt} - \overline{\mathcal{F}}_r - \overline{\mathcal{F}}_g\right)^\dagger \mathcal{F}_2' + (s\leftrightarrow t)\right] \\
&\qquad + \frac{N^2-1}{N}\Bigg[-\frac{N^4-N^2-1}{2N^2}\, f_3^\dagger(s,t,u) f_3(t,s,u) \\
&\qquad\qquad + \frac{N^4-2N^2-1}{2N^2}\left[f_3^\dagger(s,t,u) f_4(t,s,u) + (s\leftrightarrow t)\right] \\
&\qquad\qquad + \frac{3N^2+1}{2N^2}\, f_4^\dagger(s,t,u) f_4(t,s,u)\Bigg]\Bigg\}. \qquad (4.18)
\end{aligned}
$$

The infrared singular functions are given by

$$
\overline{\mathcal{IR}}_t = \frac{2}{\epsilon(2+\epsilon)}\left[\frac{1}{N}\mathrm{Bub}(s) + \frac{1}{N}\mathrm{Bub}(t) - \frac{(N^2+1)}{N}\mathrm{Bub}(u)\right], \tag{4.19}
$$

$$
\overline{\mathcal{IR}}_{nt} = \frac{2}{\epsilon(2+\epsilon)}\left[\frac{N^2-1}{N}\mathrm{Bub}(s) - \frac{1}{N}\mathrm{Bub}(t) + \frac{1}{N}\mathrm{Bub}(u)\right], \tag{4.20}
$$

which diverge as $1/\epsilon^2$. The finite renormalisation term is

$$
\overline{\mathcal{F}}_r = \beta_0\left(-\frac{1}{\epsilon} + \frac{3(1-\epsilon)}{3-2\epsilon}\mathrm{Bub}(t)\right), \tag{4.21}
$$

while the remaining finite contribution multiplying tree-level is given by

$$\overline{\mathcal{F}}_g = \frac{\epsilon\left(N^2(11+2\epsilon)+9-4\epsilon^2\right)}{2(2+\epsilon)(3-2\epsilon)N}\mathrm{Bub}(t). \tag{4.22}$$

Once again, the finite part of the crossed one loop amplitude multiplying the infrared divergent terms of the one loop amplitude generates finite functions

$$\mathcal{F}_1' = \frac{N^2-1}{2N^2}\left[\left(N^2-2\right)f_1(s,t,u)+2f_2(s,t,u)\right], \tag{4.23}$$

and

$$\mathcal{F}_2' = \frac{N^2-1}{2N^2}\left[f_1(s,t,u)-f_2(s,t,u)\right], \tag{4.24}$$

where

$$\begin{aligned} f_1(s,t,u) = {} & \frac{2u}{st}(1-2\epsilon)\left[t^2+u^2-2\epsilon(t^2+s^2)+\epsilon^2 s^2\right]\mathrm{Box}^6(s,t) \\ & +\frac{2}{st(2+\epsilon)}\Big[2u(2u-t)+\epsilon(u^2-tu-4t^2)+\epsilon^2(ts-4u^2) \\ & \qquad +\epsilon^3 us+\epsilon^4 ts\Big]\left[\mathrm{Bub}(t)-\mathrm{Bub}(s)\right], \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} f_2(s,t,u) = {} & \frac{2}{s}(1-2\epsilon)\left[2u^2-\epsilon(t^2+s^2+u^2)+3s^2\epsilon^2+s^2\epsilon^3\right]\mathrm{Box}^6(s,u) \\ & \frac{2}{ts(2+\epsilon)}\Big[6u^2-2t^2\epsilon-\epsilon^2(2t^2+5u^2+3tu)-\epsilon^3 s^2 \\ & \qquad +\epsilon^4 ts\Big]\left[\mathrm{Bub}(u)-\mathrm{Bub}(s)\right]. \end{aligned} \tag{4.26}$$

Finally the square of the finite part of the one-loop amplitude is controlled by the finite functions f_3 and f_4

$$f_3(s,t,u) = \frac{1}{s}\left\{(s^2+u^2)\mathrm{Box}^6(s,t)+(2u-s)\Big[\mathrm{Bub}(s)-\mathrm{Bub}(t)\Big]\right\}+\mathcal{O}(\epsilon), \tag{4.27}$$

and

$$f_4(s,t,u) = \frac{u}{s}\left\{2s\mathrm{Box}^6(t,u)+3\Big[\mathrm{Bub}(u)-\mathrm{Bub}(t)\Big]\right\}+\mathcal{O}(\epsilon). \tag{4.28}$$

Again, the infrared singular structure obtained by explicit expansion of $\overline{\mathcal{IR}}_t$ and $\overline{\mathcal{IR}}_{nt}$ as series in ϵ, agrees with that obtained using Catani's formalism [13]

$$\overline{\mathcal{IR}}_{t,C} = \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\left\{\frac{1}{N}\left(-\frac{\mu^2}{s}\right)^{\epsilon}+\frac{1}{N}\left(-\frac{\mu^2}{t}\right)^{\epsilon}-\frac{(N^2+1)}{N}\left(-\frac{\mu^2}{u}\right)^{\epsilon}\right\}, \tag{4.29}$$

and

$$\overline{\mathcal{IR}}_{nt,C} = \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\left\{\frac{N^2-1}{N}\left(-\frac{\mu^2}{s}\right)^{\epsilon}-\frac{1}{N}\left(-\frac{\mu^2}{t}\right)^{\epsilon}+\frac{1}{N}\left(-\frac{\mu^2}{u}\right)^{\epsilon}\right\}. \tag{4.30}$$

As before, we can rewrite Eq. (4.18) directly in terms of $\overline{\mathcal{IR}}_{t,C}$ and $\overline{\mathcal{IR}}_{nt,C}$ rather than $\overline{\mathcal{IR}}_t$ and $\overline{\mathcal{IR}}_{t,C}$ provided the finite difference is evaluated through to $\mathcal{O}(\epsilon^2)$.

5. Conclusions

In this paper we discussed the $\mathcal{O}(\alpha_s^4)$ virtual corrections to like and unlike massless quark-quark scattering formed by the interference of one-loop amplitudes with one-loop amplitudes. Our main results are Eqs. (4.1) and (4.18) where we provided analytic formulae for the $\overline{\mathrm{MS}}$-renormalised amplitudes in terms of two one-loop master integrals, the box graph in $6-2\epsilon$ dimensions and the bubble diagram in $4-2\epsilon$ dimensions. Expressions for the crossed and time reversed processes can be simply produced by inserting the expansions of the master integrals in the appropriate kinematic region. Together with the interference of two-loop diagrams with tree graphs computed in [11, 12], the one-loop square contributions given in Eqs. (4.1) and (4.18) complete the set of $2 \to 2$ virtual corrections for massless quark-quark scattering at NNLO.

Acknowledgements

C.A. acknowledges the financial support of the Greek Government and M.E.T. acknowledges financial support from CONACyT and the CVCP. We gratefully acknowledge the support of the British Council and German Academic Exchange Service under ARC project 1050. This work was supported in part by the EU Fourth Framework Programme 'Training and Mobility of Researchers', Network 'Quantum Chromodynamics and the Deep Structure of Elementary Particles', contract FMRX-CT98-0194 (DG-12-MIHT).

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